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Generalized Weierstrass representation of surfaces in $\mathbf{R}^4$ [☆]

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Abstract

In the present paper, we describe the conformal immersion of the surface into $\mathbf{R}^4$ by means of a linear system. Furthermore we prove that every regular conformal immersion of a surface into $\mathbf{R}^4$ is locally determined by the generalized Weierstrass formulae. We also give the representation of the surface with parallel mean curvature vector by solutions with the parameter of a linear system which is determined by the sinh–Laplace equation.

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Keywords: Surface; Conformal immersion; Weierstrass representation

0. Introduction

Surface theory has been intensively studied in mathematics and physics. The application of the theory to solitary wave phenomena in physics yields so-called "soliton geometry". An important branch is the Weierstrass representation of the surface in constant curvature space. The representation makes us study surfaces and their properties by means of analysis methods. A classical example of such an approach is given by the Weierstrass representation for the minimal surface in $\mathbf{R}^3$.

In $\mathbf{R}^3$ and $\mathbf{H}^3(-1)$, there are abundant results with respect to the Weierstrass representations of the surfaces with constant mean curvature (e.g. Refs [2,4,5,10]).

Generalized Weierstrass formulae for the surfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$ were given by Konopelchenko in [6,8], and received much attention (see e.g. [1,7,9]).

F. Pedit and U. Pinkall gave a coordinate-free version of the generalized Weierstrass representations for the conformal immersions of the surfaces into $\mathbf{R}^3$ and $\mathbf{R}^4$ using the theory of quaternionic line bundles in [1]. These representations are more intrinsic.

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In the present paper, we study the conformal immersion of the surface into $\mathbf{R}^4$. We give a coordinate version of the generalized Weierstrass representation for the surface in $\mathbf{R}^4$ similar to the classical expression.

The surface with parallel mean curvature vector in $\mathbf{R}^4$ has been one of the interesting subjects. There are abundant results for the surface (e.g. [11]). We study the surface by using an integrable system and find that the surface is determined by the sinh–Laplace equation.

1. 2 by 2 matrix description

Let us denote the algebra of the quaternion by $\mathbf{H}$, the multiplicative quaternion group by $\mathbf{H}_* = \mathbf{H} \setminus \{0\}$. We identify a four-dimensional Euclidean space with the quaternion $\mathbf{H}$ as follows:

$$X = (X^1, X^2, X^3, X^4) \longleftrightarrow \begin{pmatrix} X^1 - iX^4 & -(X^3 + iX^2) \\ X^3 - iX^2 & X^1 + iX^4 \end{pmatrix}. \tag{1.1}$$

And also, we still denote $\begin{pmatrix} X^1 - iX^4 & -(X^3 + iX^2) \\ X^3 - iX^2 & X^1 + iX^4 \end{pmatrix}$ by X.

Define the scalar product of $\mathbf{H}$ as

$$\langle X, X \rangle = \det X, \quad X \in \mathbf{H}. \tag{1.2}$$

or

$$\langle X, Y \rangle = \frac{1}{2}(\det(X + Y) - \det X - \det Y), \quad X, Y \in \mathbf{H}. \tag{1.3}$$

According to this scalar product, the quaternion $\mathbf{H}$ turns out to be a Euclidean space.

Let $f : D \subseteq C \to \mathbf{R}^4$ be a conformal immersed surface, and $e^{\omega} \mathrm{d}z\mathrm{d}\bar{z}$ be the first fundamental form of the surface. Then $\langle f_z, f_{\bar{z}} \rangle = \frac{1}{2} e^{\omega}$.

We denote the immersion by

$$f = \begin{pmatrix} f^1 - if^4 & -(f^3 + if^2) \\ f^3 - if^2 & f^1 + if^4 \end{pmatrix}, \tag{1.4}$$

where f^j $(j = 1, 2, 3, 4)$ are functions with real values.

Proposition 1.1. *Let $f : D \subseteq C \to \mathbf{R}^4$ be a conformal immersed surface. Then there exist $\phi_1, \phi_2 \in \mathbf{H}_*$ s.t.*

$$f_z = i\phi_2^* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \phi_1, \qquad f_{\bar{z}} = i\phi_2^* \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \phi_1. \tag{1.5}$$

Two normal vectors of the surface can be given respectively as

$$n_1 = ie^{-\frac{w}{2}} \phi_2^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \phi_1, \qquad n_2 = ie^{-\frac{w}{2}} \phi_2^* \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \phi_1, \tag{1.6}$$

where z is a conformal coordinate and $\phi_j = \begin{pmatrix} \psi_j & -\bar{\varphi}_j \\ \varphi_j & \bar{\psi}_j \end{pmatrix}$ $(j = 1, 2)$.

Proof. Let u be a conformal coordinate of the surface. Then

$$\det f_u = \det \begin{pmatrix} f_u^1 - if_u^4 & -(f_u^3 + if_u^2) \\ f_u^3 - if_u^2 & f_u^1 + if_u^4 \end{pmatrix} = (f_u^1)^2 + (f_u^2)^2 + (f_u^3)^2 + (f_u^4)^2 = 0.$$

This shows that the rank of the matrix f_u must be one. We have the matrix decomposition as follows

$$\begin{aligned} f_u &= \begin{pmatrix} f_u^1 - if_u^4 & -(f_u^3 + if_u^2) \\ f_u^3 - if_u^2 & f_u^1 + if_u^4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \end{pmatrix} \\ &= \begin{pmatrix} -b_1 & \bar{b_2} \\ -b_2 & -\bar{b_1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\bar{c_2} & \bar{c_1} \\ -c_1 & -c_2 \end{pmatrix}. \end{aligned} \tag{1.7}$$

Then

$$f_u^1 - i f_u^4 = b_1 c_1, \qquad -(f_u^3 + i f_u^2) = b_1 c_2,$$
$$f_u^3 - i f_u^2 = b_2 c_1, \qquad f_u^1 + i f_u^4 = b_2 c_2.$$

Taking the conjugation, we get

$$f_{\bar u}^1 - i f_{\bar u}^4 = \bar{b_2}\bar{c_2}, \qquad -(f_{\bar u}^3 + i f_{\bar u}^2) = -\bar{b_2}\bar{c_1},$$
$$f_{\bar u}^3 - i f_{\bar u}^2 = -\bar{b_1}\bar{c_2}, \qquad f_{\bar u}^1 + i f_{\bar u}^4 = \bar{b_1}\bar{c_1}.$$

Hence

$$f_{\bar u} = \begin{pmatrix} f_{\bar u}^1 - i f_{\bar u}^4 & -(f_{\bar u}^3 + i f_{\bar u}^2) \\ f_{\bar u}^3 - i f_{\bar u}^2 & f_{\bar u}^1 + i f_{\bar u}^4 \end{pmatrix} = \begin{pmatrix} \bar{b}_2 \\ -\bar{b}_1 \end{pmatrix} \begin{pmatrix} \bar{c}_2 & -\bar{c}_1 \end{pmatrix}$$
$$= -\begin{pmatrix} -b_1 & \bar{b}_2 \\ -b_2 & -\bar{b}_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\bar{c_2} & \bar{c_1} \\ -c_1 & -c_2 \end{pmatrix}.$$

Define $z = -iu$, $\phi_1 = \begin{pmatrix} -\bar{c_2} & \bar{c_1} \\ -c_1 & -c_2 \end{pmatrix}$, $\phi_2 = \begin{pmatrix} -\bar{b}_1 & -\bar{b}_2 \\ b_2 & -b_1 \end{pmatrix}$. Then

$$f_z = i f_u = i\phi_2^* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \phi_1, \qquad f_{\bar z} = -i f_{\bar u} = i\phi_2^* \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \phi_1.$$

Straightforwardly we check that n_1, n_2 given by (1.6) are the unit normal frame fields of the surface. □

Define $Q_j = \langle f_{zz}, n_j \rangle$ and $H_j = 2e^{-\omega} \langle f_{z\bar z}, n_j \rangle$. We call Q_j the Hopf differential and $\frac{1}{2} H_1 e_1 + \frac{1}{2} H_2 e_2$ the mean curvature vector of the surface, where $e_1 = e^{-\frac{\omega}{2}} f_x$, $e_2 = e^{-\frac{\omega}{2}} f_y$.

From these definitions, we can directly verify the following proposition.

Proposition 1.2. *Define $c = \langle n_{1z}, n_2 \rangle = -\langle n_1, n_{2z} \rangle$. Then*

$$f_{zz} = \omega_z f_z + Q_1 n_1 + Q_2 n_2 = i\phi_2^* \begin{pmatrix} (Q_1 + iQ_2) e^{-\frac{\omega}{2}} & \omega_z \\ 0 & -(Q_1 - iQ_2) e^{-\frac{\omega}{2}} \end{pmatrix} \phi_1,$$

$$f_{z\bar z} = \frac{1}{2} H_1 e^{\omega} n_1 + \frac{1}{2} H_2 e^{\omega} n_2 = i\phi_2^* \begin{pmatrix} \frac{1}{2}(H_1 + iH_2) e^{\frac{\omega}{2}} & 0 \\ 0 & -\frac{1}{2}(H_1 - iH_2) e^{\frac{\omega}{2}} \end{pmatrix} \phi_1,$$

$$f_{\bar z\bar z} = \omega_{\bar z} f_{\bar z} + \bar{Q}_1 n_1 + \bar{Q}_2 n_2 = i\phi_2^* \begin{pmatrix} (\bar{Q}_1 + i\bar{Q}_2) e^{-\frac{\omega}{2}} & 0 \\ \omega_{\bar z} & -(\bar{Q}_1 - i\bar{Q}_2) e^{-\frac{\omega}{2}} \end{pmatrix} \phi_1,$$

$$n_{1z} = -H_1 f_z - 2Q_1 e^{-\omega} f_{\bar z} + c n_2 = i e^{-\frac{\omega}{2}} \phi_2^* \begin{pmatrix} ci & -H_1 e^{\frac{\omega}{2}} \\ -2Q_1 e^{-\frac{\omega}{2}} & ci \end{pmatrix} \phi_1,$$

$$n_{1\bar z} = -2\bar{Q}_1 e^{-\omega} f_z - H_1 f_{\bar z} + \bar{c} n_2 = i e^{-\frac{\omega}{2}} \phi_2^* \begin{pmatrix} \bar{c}i & -2\bar{Q}_1 e^{-\frac{\omega}{2}} \\ -H_1 e^{\frac{\omega}{2}} & \bar{c}i \end{pmatrix} \phi_1,$$

$$n_{2z} = -H_2 f_z - 2Q_2 e^{-\omega} f_{\bar z} - c n_1 = i e^{-\frac{\omega}{2}} \phi_2^* \begin{pmatrix} -c & -H_2 e^{\frac{\omega}{2}} \\ -2Q_2 e^{-\frac{\omega}{2}} & c \end{pmatrix} \phi_1,$$

$$n_{2\bar z} = -2\bar{Q}_2 e^{-\omega} f_z - H_2 f_{\bar z} - \bar{c} n_1 = i e^{-\frac{\omega}{2}} \phi_2^* \begin{pmatrix} -\bar{c} & -2\bar{Q}_2 e^{-\frac{\omega}{2}} \\ -H_2 e^{\frac{\omega}{2}} & \bar{c} \end{pmatrix} \phi_1.$$

Define

$$\phi_2^{*-1} \phi_{2\bar z}^* = U_2^*, \qquad \phi_2^{*-1} \phi_{2z}^* = V_2^*, \qquad \phi_{1z} \phi_1^{-1} = U_1, \qquad \phi_{1\bar z} \phi_1^{-1} = V_1. \tag{1.8}$$

Next we shall express U_1, U_2, V_1, V_2 in terms of ω, Q_1, Q_2, H_1 and H_2. From the integrable condition of (1.5), we have

$$U_2^*\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}V_1=V_2^*\begin{pmatrix}0&0\\1&0\end{pmatrix}+\begin{pmatrix}0&0\\1&0\end{pmatrix}U_1. \tag{1.9}$$

Define $e^{\omega_1}I := (|\psi_1|^2+|\varphi_1|^2)I=\phi_1\phi_1^*$, $e^{\omega_2}I := (|\psi_2|^2+|\varphi_2|^2)I=\phi_2\phi_2^*$. Then

$$\omega=\omega_1+\omega_2,\qquad U_1+V_1^*=\omega_{1z},\qquad U_2+V_2^*=\omega_{2z}. \tag{1.10}$$

From (1.5), we get

$$\begin{cases} f_{zz}=i\phi_2^*\left(V_2^*\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}U_1\right)\phi_1,\\ f_{z\bar z}=i\phi_2^*\left(U_2^*\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}V_1\right)\phi_1,\\ f_{\bar z\bar z}=i\phi_2^*\left(U_2^*\begin{pmatrix}0&0\\1&0\end{pmatrix}+\begin{pmatrix}0&0\\1&0\end{pmatrix}V_1\right)\phi_1. \end{cases} \tag{1.11}$$

And

$$n_{1z}=ie^{-\frac{\omega}{2}}\phi_2^*\left(-\frac{1}{2}\omega_z\begin{pmatrix}1&0\\0&-1\end{pmatrix}+V_2^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}U_1\right)\phi_1. \tag{1.12}$$

Combining Proposition 1.2 with (1.10) and (1.11), we obtain

$$\begin{cases} V_2^*\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}U_1=\begin{pmatrix}(Q_1+iQ_2)e^{-\frac{\omega}{2}}&\omega_z\\0&-(Q_1-iQ_2)e^{-\frac{\omega}{2}}\end{pmatrix},\\ V_2^*\begin{pmatrix}0&0\\1&0\end{pmatrix}+\begin{pmatrix}0&0\\1&0\end{pmatrix}U_1=\begin{pmatrix}\frac{1}{2}(H_1+iH_2)e^{\frac{\omega}{2}}&0\\0&-\frac{1}{2}(H_1-iH_2)e^{\frac{\omega}{2}}\end{pmatrix}. \end{cases} \tag{1.13}$$

Let $U_1=(u_{ij})$, $V_2=(v_{ij})$. From (1.13), we know

$$\begin{cases} U_1=\begin{pmatrix}u_{11}&-\frac{1}{2}(H_1-iH_2)e^{\frac{\omega}{2}}\\(Q_1+iQ_2)e^{-\frac{\omega}{2}}&\omega_z-\bar v_{11}\end{pmatrix},\\ V_2=\begin{pmatrix}v_{11}&-(\bar Q_1+i\bar Q_2)e^{-\frac{\omega}{2}}\\(H_1-iH_2)e^{\frac{\omega}{2}}&-\bar u_{11}\end{pmatrix}. \end{cases} \tag{1.14}$$

Since $U_2=\omega_{2z}I-V_2^*$ and $V_1^*=\omega_{1z}I-U_1$, we have

$$\begin{cases} U_2=\begin{pmatrix}\omega_{2z}-\bar v_{11}&-\frac{1}{2}(H_1+iH_2)e^{\frac{\omega}{2}}\\(Q_1-iQ_2)e^{-\frac{\omega}{2}}&\omega_{2z}+u_{11}\end{pmatrix},\\ V_1=\begin{pmatrix}\omega_{1\bar z}-\bar u_{11}&-(\bar Q_1-i\bar Q_2)e^{-\frac{\omega}{2}}\\\frac{1}{2}(H_1+iH_2)e^{\frac{\omega}{2}}&-\omega_{2\bar z}+v_{11}\end{pmatrix}. \end{cases} \tag{1.15}$$

From $\phi_{1z}=U_1\phi_1$ and $\phi_{1\bar z}=V_1\phi_1$, we find

$$\begin{cases} \psi_{1z}=u_{11}\psi_1-\frac{1}{2}(H_1-iH_2)e^{\frac{\omega}{2}}\varphi_1,\\ \bar\psi_{1\bar z}=-\frac{1}{2}(H_1+iH_2)e^{\frac{\omega}{2}}\bar\varphi_1+(-\omega_{2\bar z}+v_{11})\bar\psi_1, \end{cases}$$

and

$$\begin{cases}\varphi_{1z} = (Q_1 + iQ_2)e^{-\frac{\omega}{2}}\psi_1 + (\omega_z - \bar{v}_{11})\varphi_1,\\ \bar{\varphi}_{1\bar{z}} = (\omega_{1\bar{z}} - \bar{u}_{11})\bar{\varphi}_1 + (\bar{Q}_1 - i\bar{Q}_2)e^{-\frac{\omega}{2}}\bar{\psi}_1.\end{cases}$$

So

$$u_{11} = -\omega_{2z} + \bar{v}_{11}. \tag{1.16}$$

From (1.12) and Proposition 1.2, we get

$$-\frac{1}{2}\omega_z + \bar{v}_{11} + u_{11} = ic. \tag{1.17}$$

(1.16) and (1.17) yield

$$\begin{cases}\bar{v}_{11} = \frac{1}{2}ic + \frac{1}{4}\omega_{1z} + \frac{3}{4}\omega_{2z},\\ u_{11} = \frac{1}{2}ic + \frac{1}{4}\omega_{1z} - \frac{1}{4}\omega_{2z}.\end{cases} \tag{1.18}$$

From (1.12), (1.14) and (1.15) we obtain the following two linear systems.

$$\begin{cases}\phi_{1z} = \begin{pmatrix}\frac{i}{2}c + \frac{1}{4}(\omega_1 - \omega_2)_z & -\frac{1}{2}(H_1 - iH_2)e^{\frac{\omega}{2}},\\ (Q_1 + iQ_2)e^{-\frac{\omega}{2}} & \frac{3}{4}\omega_{1z} + \frac{1}{4}\omega_{2z} - \frac{1}{2}ic\end{pmatrix}\phi_1,\\ \phi_{1\bar{z}} = \begin{pmatrix}\frac{i}{2}\bar{c} + \frac{3}{4}\omega_{1\bar{z}} + \frac{1}{4}\omega_{2\bar{z}} & -(\bar{Q}_1 - i\bar{Q}_2)e^{-\frac{\omega}{2}}\\ \frac{1}{2}(H_1 + iH_2)e^{\frac{\omega}{2}} & -\frac{1}{2}i\bar{c} + \frac{1}{4}(\omega_1 - \omega_2)_{\bar{z}}\end{pmatrix}\phi_1,\end{cases} \tag{1.19}$$

$$\begin{cases}\phi_{2z} = \begin{pmatrix}-\frac{i}{2}c - \frac{1}{4}(\omega_1 - \omega_2)_z & -\frac{1}{2}(H_1 + iH_2)e^{\frac{\omega}{2}},\\ (Q_1 - iQ_2)e^{-\frac{\omega}{2}} & \frac{1}{4}\omega_{1z} + \frac{3}{4}\omega_{2z} + \frac{1}{2}ic\end{pmatrix}\phi_2,\\ \phi_{2\bar{z}} = \begin{pmatrix}-\frac{i}{2}\bar{c} + \frac{1}{4}\omega_{1\bar{z}} + \frac{3}{4}\omega_{2\bar{z}} & -(\bar{Q}_1 + i\bar{Q}_2)e^{-\frac{\omega}{2}},\\ \frac{1}{2}(H_1 - iH_2)e^{\frac{\omega}{2}} & \frac{1}{2}i\bar{c} - \frac{1}{4}(\omega_1 - \omega_2)_{\bar{z}}\end{pmatrix}\phi_2.\end{cases} \tag{1.20}$$

Note that the equation $\theta_z = -\frac{c}{2}$ always is solvable. Let θ be a solution of this equation. By the gauge transformations

$$\begin{cases}\tilde{\phi}_1 = e^{-\frac{1}{4}(\omega_1 - \omega_2)}\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{-i\bar{\theta}}\end{pmatrix}\phi_1,\\ \tilde{\phi}_2 = e^{\frac{1}{4}(\omega_1 - \omega_2)}\begin{pmatrix}e^{-i\theta} & 0\\ 0 & e^{i\bar{\theta}}\end{pmatrix}\phi_2,\end{cases} \tag{1.21}$$

we can obtain the following

Theorem 1.3. *Under the isomorphism (1.1), the moving frame $f_z, f_{\bar{z}}, n_1, n_2$ of the conformally immersed surface in $\mathbf{R}^4$ are*

$$f_z = i\phi_2^*\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\phi_1, \qquad f_{\bar{z}} = i\phi_2^*\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\phi_1 \tag{1.22}$$

$$\begin{cases} n_1 = ie^{-\frac{\omega}{2}}\phi_2^* \begin{pmatrix} e^{-i(\theta+\bar{\theta})} & 0 \\ 0 & -e^{i(\theta+\bar{\theta})} \end{pmatrix} \phi_1, \\ n_2 = ie^{-\frac{\omega}{2}}\phi_2^* \begin{pmatrix} ie^{-i(\theta+\bar{\theta})} & 0 \\ 0 & ie^{i(\theta+\bar{\theta})} \end{pmatrix} \phi_1, \end{cases} \tag{1.23}$$

where $\phi_1, \phi_2 \in \mathbf{H}_*$ *satisfy*

$$\begin{cases} \phi_{1z} = \begin{pmatrix} 0 & -\frac{1}{2}(H_1 - iH_2)e^{\frac{\omega}{2}+i(\theta+\bar{\theta})} \\ (Q_1 + iQ_2)e^{-\frac{\omega}{2}-i(\theta+\bar{\theta})} & \frac{1}{2}\omega_z - i\bar{\theta}_z - \frac{1}{2}ic \end{pmatrix} \phi_1, \\ \phi_{1\bar{z}} = \begin{pmatrix} \frac{1}{2}\omega_{\bar{z}} + i\theta_{\bar{z}} + \frac{1}{2}i\bar{c} & -(\bar{Q}_1 - i\bar{Q}_2)e^{-\frac{\omega}{2}+i(\theta+\bar{\theta})} \\ \frac{1}{2}(H_1 + iH_2)e^{\frac{\omega}{2}-i(\theta+\bar{\theta})} & 0 \end{pmatrix} \phi_1, \end{cases} \tag{1.24}$$

$$\begin{cases} \phi_{2z} = \begin{pmatrix} 0 & -\frac{1}{2}(H_1 + iH_2)e^{\frac{\omega}{2}-i(\theta+\bar{\theta})} \\ (Q_1 - iQ_2)e^{-\frac{\omega}{2}+i(\theta+\bar{\theta})} & \frac{1}{2}\omega_z + i\bar{\theta}_z + \frac{1}{2}ic \end{pmatrix} \phi_2, \\ \phi_{2\bar{z}} = \begin{pmatrix} \frac{1}{2}\omega_{\bar{z}} - i\theta_{\bar{z}} - \frac{1}{2}i\bar{c} & -(\bar{Q}_1 + i\bar{Q}_2)e^{-\frac{\omega}{2}-i(\theta+\bar{\theta})} \\ \frac{1}{2}(H_1 - iH_2)e^{\frac{\omega}{2}+i(\theta+\bar{\theta})} & 0 \end{pmatrix} \phi_2. \end{cases} \tag{1.25}$$

The compatible conditions are

Gauss equation $\quad \omega_{z\bar{z}} + \frac{1}{2}|H|^2 e^{\omega} - 2(|Q_1|^2 + |Q_2|^2)e^{-\omega} = 0. \quad (1.26)$

Ricci equation $\quad c_{\bar{z}} - \bar{c}_z = 2(Q_1\bar{Q}_2 - \bar{Q}_1 Q_2). \quad (1.27)$

Codazzi equation (1.28)

$$Q_{1\bar{z}} - \bar{c}Q_2 = \frac{1}{2}e^{\omega}(H_{1z} - cH_2), \qquad Q_{2\bar{z}} + \bar{c}Q_1 = \frac{1}{2}e^{\omega}(H_{2z} + cH_1),$$

$$\bar{Q}_{1z} - c\bar{Q}_2 = \frac{1}{2}e^{\omega}(H_{1\bar{z}} - \bar{c}H_2), \qquad \bar{Q}_{2z} + c\bar{Q}_1 = \frac{1}{2}e^{\omega}(H_{2\bar{z}} + \bar{c}H_1).$$

And c, θ *satisfy*

$$\theta_z = -\frac{1}{2}c. \tag{1.29}$$

Proof. Straightforwardly check by substituting the gauge transformation (1.21) into (1.5), (1.19) and (1.20) and rewrite $\tilde{\phi}_1, \tilde{\phi}_2$ as ϕ_1, ϕ_2. □

Also from the gauge transformation, we get

$$f_{z\bar{z}} = \frac{i}{2}e^{\frac{\omega}{2}}\phi_2^* \begin{pmatrix} (H_1 + iH_2)e^{-i(\theta+\bar{\theta})} & 0 \\ 0 & -(H_1 - iH_2)e^{i(\theta+\bar{\theta})} \end{pmatrix} \phi_1, \tag{1.30}$$

For convenience, sometimes we define $\theta + \bar{\theta} = u$ (a real function).

2. Weierstrass formulae

The generalized Weierstrass formulae of the surfaces in $\mathbf{R}^3$ with coordinates X^1, X^2, X^3 are of following form (e.g. Ref. [6])

$$X^1 + iX^2 = i\int_{\Gamma}(\bar{\varphi}^2 \mathrm{d}z - \bar{\psi}^2 \mathrm{d}\bar{z}),\ X^1 - iX^2 = i\int_{\Gamma}(\psi^2 \mathrm{d}z - \varphi^2 \mathrm{d}\bar{z}),$$

$$X^3 = \int_\Gamma (\bar{\psi}\varphi \mathrm{d}z + \bar{\varphi}\psi \mathrm{d}\bar{z}),$$

where Γ is a contour in $\mathbf{C}$, and φ, ψ satisfy

$$\varphi_z = p\psi, \qquad \psi_{\bar{z}} = -p\varphi.$$

The formulae define a conformal immersed surface in $\mathbf{R}^3$. Vice versa, every regular conformal immersed surface in $\mathbf{R}^3$ is locally defined by the above formulae [9].

The generalized Weierstrass formulae of the surfaces in $\mathbf{R}^4$ with coordinates X^1, X^2, X^3, X^4 were put forward for the first time by Konopelchenko [8]. The formulae are of the following form.

$$X^1 + iX^2 = \int_\Gamma (-\varphi_1\varphi_2 \mathrm{d}z + \psi_1\psi_2 \mathrm{d}\bar{z}), \qquad X^3 + iX^4 = \int_\Gamma (\varphi_1\bar{\psi}_2 \mathrm{d}z + \bar{\varphi}_2\psi_1 \mathrm{d}\bar{z}),$$
$$X^1 - iX^2 = \int_\Gamma (\bar{\psi}_1\bar{\psi}_2 \mathrm{d}z - \bar{\varphi}_1\bar{\varphi}_2 \mathrm{d}\bar{z}), \qquad X^3 - iX^4 = \int_\Gamma (\varphi_2\bar{\psi}_1 \mathrm{d}z + \psi_2\bar{\varphi}_1 \mathrm{d}\bar{z}),$$

where

$$\psi_{1z} = p\varphi_1, \qquad \psi_{2z} = \bar{p}\varphi_2,$$
$$\varphi_{1\bar{z}} = -\bar{p}\psi_1, \qquad \varphi_{2\bar{z}} = -p\psi_2.$$

and $\psi_\alpha, \varphi_\alpha$ $(\alpha = 1, 2)$, p are complex-valued functions of $z, \bar{z}$, Γ is a contour in $\mathbf{C}$.

However B.G. Konopelchenko did not show whether a conformal surface immersed into $\mathbf{R}^4$ is of this representation locally.

Pedit and Pinkall study the conformal immersions of the surfaces into $\mathbf{R}^3$ and $\mathbf{R}^4$ using the quaternionic value function theory in [1]. In the case of $\mathbf{R}^4$, they obtained the following coordinate-free version of the generalized Weierstrass representation.

Theorem A (*F. Pedit and U. Pinkall*). *Let $f : M \to \mathbf{H}$ be a conformal immersion. Then there exist paired holomorphic quaternionic line bundles $L, \tilde{L}$ and nowhere vanishing sections $\psi \in H^0(L)$, $\phi \in H^0(\tilde{L})$ such that*

$$\mathrm{d}f = (\psi, \phi).$$

$L, \tilde{L}, \psi$ and ϕ are uniquely determined by f up to isomorphism. And ψ and ϕ satisfy $(\bar{\partial} + Q)\psi = 0$, $(\bar{\partial} + \tilde{Q})\phi = 0$, where $Q, \tilde{Q}$ are Hopf fields.

The classical Weierstrass representation is given by the coordinates. We hope to get the coordinate version of the generalized Weierstrass representation in the case of $\mathbf{R}^4$. Using the conclusions of Section 1, we can obtain the following Weierstrass representation of the surfaces immersed conformally into $\mathbf{R}^4$. Comparing with Theorem A, our conclusion is given in coordinate form.

Theorem 2.4. *Assume that $f : D \subseteq C \to \mathbf{R}^4$ is a regular conformal immersion of surface M into $\mathbf{R}^4$ with conformal coordinates z. Let $e^\omega \mathrm{d}z\mathrm{d}\bar{z}$ be the metric of M. Then f can be locally expressed by*

$$\begin{aligned} f^4 + if^1 &= \int_\Gamma (\varphi_1\bar{\psi}_2 \mathrm{d}z + \bar{\varphi}_2\psi_1 \mathrm{d}\bar{z}), \qquad & f^2 - if^3 &= \int_\Gamma (-\bar{\psi}_1\bar{\psi}_2 \mathrm{d}z + \bar{\varphi}_1\bar{\varphi}_2 \mathrm{d}\bar{z}), \\ f^2 + if^3 &= \int_\Gamma (\varphi_1\varphi_2 \mathrm{d}z - \psi_1\psi_2 \mathrm{d}\bar{z}), \qquad & f^4 - if^1 &= -\int_\Gamma (\varphi_2\bar{\psi}_1 \mathrm{d}z + \psi_2\bar{\varphi}_1 \mathrm{d}\bar{z}), \end{aligned} \tag{2.1}$$

where $\psi_\alpha, \varphi_\alpha$ $(\alpha = 1, 2)$ satisfy

$$\begin{cases} \psi_{1z} = -\dfrac{1}{2}(H_1 - iH_2)e^{\frac{\omega}{2}+iu}\varphi_1, \\ \varphi_{1\bar{z}} = \dfrac{1}{2}(H_1 + iH_2)e^{\frac{\omega}{2}-iu}\psi_1, \end{cases} \qquad \begin{cases} \psi_{2z} = -\dfrac{1}{2}(H_1 + iH_2)e^{\frac{\omega}{2}-iu}\varphi_2, \\ \varphi_{2\bar{z}} = \dfrac{1}{2}(H_1 - iH_2)e^{\frac{\omega}{2}+iu}\psi_2. \end{cases} \tag{2.2}$$

Proof. Let

$$\phi_1 = \begin{pmatrix} \psi_1 & -\bar{\varphi}_1 \\ \varphi_1 & \bar{\psi}_1 \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} \psi_2 & -\bar{\varphi}_2 \\ \varphi_2 & \bar{\psi}_2 \end{pmatrix}.$$

(1.24) and (1.25) show that (2.2) holds.

From (1.22), we have

$$\begin{aligned} \mathrm{d}f &= f_z \mathrm{d}z + f_{\bar{z}} \mathrm{d}\bar{z} \\ &= i \begin{pmatrix} \varphi_1 \bar{\psi}_2 \mathrm{d}z + \bar{\varphi}_2 \psi_1 \mathrm{d}\bar{z} & \bar{\psi}_1 \bar{\psi}_2 \mathrm{d}z - \bar{\varphi}_1 \bar{\varphi}_2 \mathrm{d}\bar{z} \\ -\varphi_1 \varphi_2 \mathrm{d}z + \psi_1 \psi_2 \mathrm{d}\bar{z} & -\varphi_2 \bar{\psi}_1 \mathrm{d}z - \bar{\varphi}_1 \psi_2 \mathrm{d}\bar{z} \end{pmatrix}. \end{aligned}$$

Combining with (1.4)

$$\mathrm{d}f = \begin{pmatrix} \mathrm{d}(f^1 - if^4) & -\mathrm{d}(f^3 + if^2) \\ \mathrm{d}(f^3 - if^2) & \mathrm{d}(f^1 + if^4) \end{pmatrix},$$

we have (2.1). □

Two normal vectors n_1 and n_2 satisfy

$$\begin{aligned} n_1 &= ie^{-\frac{\omega}{2}} \phi_2^* \begin{pmatrix} e^{-iu} & 0 \\ 0 & -e^{iu} \end{pmatrix} \phi_1 \\ &= ie^{-\frac{\omega}{2}} \begin{pmatrix} \psi_1 \bar{\psi}_2 e^{-iu} - \varphi_1 \bar{\varphi}_2 e^{iu} & -\bar{\varphi}_1 \bar{\psi}_2 e^{-iu} - \bar{\psi}_1 \bar{\varphi}_2 e^{iu} \\ -\psi_1 \varphi_2 e^{-iu} - \varphi_1 \psi_2 e^{iu} & \bar{\varphi}_1 \varphi_2 e^{-iu} - \bar{\psi}_1 \psi_2 e^{iu} \end{pmatrix}, \\ n_2 &= ie^{-\frac{\omega}{2}} \phi_2^* \begin{pmatrix} ie^{-iu} & 0 \\ 0 & ie^{iu} \end{pmatrix} \phi_1 \\ &= -e^{-\frac{\omega}{2}} \begin{pmatrix} \psi_1 \bar{\psi}_2 e^{-iu} + \varphi_1 \bar{\varphi}_2 e^{iu} & -\bar{\varphi}_1 \bar{\psi}_2 e^{-iu} + \bar{\psi}_1 \bar{\varphi}_2 e^{iu} \\ -\psi_1 \varphi_2 e^{-iu} + \varphi_1 \psi_2 e^{iu} & \bar{\varphi}_1 \varphi_2 e^{-iu} + \bar{\psi}_1 \psi_2 e^{iu} \end{pmatrix}. \end{aligned}$$

According to the (1.1), we can get the components of n_1 and n_2.

3. The surface with parallel mean curvature vector in $\mathbf{R}^4$

The surface with parallel mean curvature vector in $\mathbf{R}^n$ has been classed by Yau [11] as follows.

Theorem B. *Let M^2 be a surface with parallel mean curvature vector in a constant curved manifold* **N**. *Then either M^2 is a minimal surface of umbilical hypersurface of* **N** *or M^2 lies in a three-dimensional umbilical submanifold of* **N** *with constant mean curvature.*

We shall study the surfaces with parallel mean curvature vector in $\mathbf{R}^4$ by using the method of the integrable system. Define $e_1 = e^{-\frac{\omega}{2}} f_x,\ e_2 = e^{-\frac{\omega}{2}} f_y,\ e_3 = n_1,\ e_4 = n_2$ given as in Theorem 1.3. Then

$$f_z = \frac{1}{2} e^{\frac{\omega}{2}} (e_1 - ie_2), \qquad f_{\bar{z}} = \frac{1}{2} e^{\frac{\omega}{2}} (e_1 + ie_2), \tag{3.1}$$

or

$$e_1 = e^{-\frac{\omega}{2}} (f_z + f_{\bar{z}}), \qquad e_2 = ie^{-\frac{\omega}{2}} (f_z - f_{\bar{z}}). \tag{3.2}$$

$$\begin{aligned} \mathrm{d}e_3 &= n_{1z} \mathrm{d}z + n_{1\bar{z}} \mathrm{d}\bar{z} \\ &= (-H_1 \mathrm{d}z - 2\overline{Q}_1 e^{-\omega} \mathrm{d}\bar{z}) f_z + (-2Q_1 e^{-\omega} \mathrm{d}z - H_1 \mathrm{d}\bar{z}) f_{\bar{z}} + (c\mathrm{d}z + \bar{c}\mathrm{d}\bar{z}) e_4. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathrm{d}e_3 &= \omega_{13} e_1 + \omega_{23} e_2 + \omega_{43} e_4 \\ &= -(\omega_{31} + i\omega_{32}) e^{-\frac{\omega}{2}} f_z - (\omega_{31} - i\omega_{32}) e^{-\frac{\omega}{2}} f_{\bar{z}} - \omega_{34} e_4. \end{aligned}$$

These show that

$$\begin{cases} H_1 \mathrm{d}z + 2\overline{Q}_1 e^{-\omega} \mathrm{d}\overline{z} = (\omega_{31} + i\omega_{32}) e^{-\frac{\omega}{2}}, \\ 2Q_1 e^{-\omega} \mathrm{d}z + H_1 \mathrm{d}\overline{z} = (\omega_{31} - i\omega_{32}) e^{-\frac{\omega}{2}}, \\ c\mathrm{d}z + \overline{c}\mathrm{d}\overline{z} = -\omega_{34}. \end{cases} \tag{3.3}$$

Let $\omega_{31} = \sum h^3_{1j}\omega_j$, $\omega_{32} = \sum h^3_{2j}\omega_j$. From $\omega_1 = e^{\frac{\omega}{2}}\mathrm{d}x$, $\omega_2 = e^{\frac{\omega}{2}}\mathrm{d}y$ and (3.3), we have

$$H_1 = \frac{1}{2}(h^3_{11} + h^3_{22}), \qquad \overline{Q}_1 e^{-\omega} = \frac{1}{4}(h^3_{11} - h^3_{22}) + \frac{i}{2}h^3_{12}. \tag{3.4}$$

In the same way, we get

$$H_2 = \frac{1}{2}(h^4_{11} + h^4_{22}), \qquad \overline{Q}_2 e^{-\omega} = \frac{1}{4}(h^4_{11} - h^4_{22}) + \frac{i}{2}h^4_{12}, \tag{3.5}$$

where $\omega_{41} = \sum h^4_{1j}\omega_j,\ \omega_{42} = \sum h^4_{2j}\omega_j$.

We denote the mean curvature vector of the surface by $\mathbf{H}$. Then $\mathbf{H} = e^{-\omega} f_{z\overline{z}} = \frac{1}{2}H_1 e_3 + \frac{1}{2}H_2 e_4$. Define the covariant differentiation of e_3, e_4 in the normal bundle of M^2 by

$$D^{\perp}e_3 = -\omega_{34}e_4, \qquad D^{\perp}e_4 = \omega_{34}e_3.$$

We have

$$D^{\perp}\mathbf{H} = \frac{1}{2}\left[(\mathrm{d}H_1 + H_2\omega_{34})e_3 + (\mathrm{d}H_2 - H_1\omega_{34}e_4)\right].$$

Hence, $\mathbf{H}$ is parallel in the normal bundle of the surface in $\mathbf{R}^4$ if and only if $\mathrm{d}H_1 + H_2\omega_{34} = 0,\ \mathrm{d}H_2 - H_1\omega_{34} = 0$. That is

$$\begin{cases} H_{1z} - cH_2 = 0, \\ H_{1\overline{z}} - \overline{c}H_2 = 0, \end{cases} \qquad \begin{cases} H_{2z} + cH_1 = 0, \\ H_{2\overline{z}} + \overline{c}H_1 = 0. \end{cases} \tag{3.6}$$

It is easy to establish $|\mathbf{H}| = const.$ on the surface.

In the following case, we assume $|\mathbf{H}| \neq 0$.

Define

$$H_1 = |\mathbf{H}|\cos\varphi, \qquad H_2 = |\mathbf{H}|\sin\varphi, \qquad H = H_1 + iH_2 = |H|e^{i\varphi}. \tag{3.7}$$

We get

$$\mathrm{d}H_1 = -|H|\sin\varphi\mathrm{d}\varphi, \qquad \mathrm{d}H_2 = |H|\cos\varphi\mathrm{d}\varphi. \tag{3.8}$$

Together with $\mathrm{d}H_1 - H_2\omega_{34} = 0,\ \mathrm{d}H_2 + H_1\omega_{34} = 0$, this gives

$$\omega_{34} = -\mathrm{d}\varphi. \tag{3.9}$$

Furthermore

$$\varphi_z = -c, \qquad \varphi_{\overline{z}} = -\overline{c}. \tag{3.10}$$

From (1.28)

$$(Q_1 + iQ_2)_{\overline{z}} = -i\overline{c}(Q_1 + iQ_2) = i\varphi_{\overline{z}}(Q_1 + iQ_2),$$
$$(Q_1 - iQ_2)_{\overline{z}} = i\overline{c}(Q_1 - iQ_2) = -i\varphi_{\overline{z}}(Q_1 - iQ_2).$$

Hence

$$Q_1 + iQ_2 = \xi_1(z)e^{i\varphi}, \qquad Q_1 - iQ_2 = \xi_2(z)e^{-i\varphi}, \tag{3.11}$$

where ξ_1 and ξ_2 are holomorphic functions.

From (3.7) and (3.11), we get

$$\begin{cases} Q_1 + iQ_2 = \xi_1 e^{i\varphi} = \dfrac{1}{|H|}(H_1 + iH_2)\xi_1, \\ Q_1 - iQ_2 = \xi_2 e^{-i\varphi} = \dfrac{1}{|H|}(H_1 - iH_2)\xi_2. \end{cases} \tag{3.12}$$

Notice that $2(Q_1\overline{Q}_2 - \overline{Q}_1 Q_2) = c_{\bar{z}} - \bar{c}_z = \varphi_{\bar{z}z} - \varphi_{z\bar{z}} = 0$ and $|Q_1 + iQ_2|^2 = |Q_1|^2 + |Q_2|^2 = |Q_1 - iQ_2|^2$. We have $|\xi_1(z)| = |\xi_2(z)|$. Let $\xi_2(z) = \xi_1(z)e^{2it}$, where $t(z, \bar{z})$ is a real function. And since ξ_1 and ξ_2 are analytic, we know that t must be a constant function.

Define $\xi(z) = \xi_2(z)e^{-it} = \xi_1(z)e^{it}$. We get from (3.12)

$$\begin{cases} Q_1 + iQ_2 = \dfrac{H_1 + iH_2}{|H|} e^{-it}\xi(z), \\ Q_1 - iQ_2 = \dfrac{H_1 - iH_2}{|H|} e^{it}\xi(z). \end{cases} \tag{3.13}$$

$(Q_1 + iQ_2)\mathrm{d}z^2$ and $(Q_1 - iQ_2)\mathrm{d}z^2$ are invariant under the coordinate transformation $\mathrm{d}\widetilde{z} = \frac{1}{|H|}\xi^{\frac{1}{2}}\mathrm{d}z$. Hence we choose coordinates z such that

$$\begin{cases} Q_1 + iQ_2 = (H_1 + iH_2)e^{-it} \\ Q_1 - iQ_2 = (H_1 - iH_2)e^{it}. \end{cases} \tag{3.14}$$

Because $\theta = \frac{1}{2}\varphi$ is the solution of equation $\theta_z = -\frac{1}{2}c$, we take $\theta = \frac{1}{2}\varphi$ in (1.24) and (1.25). Then $\theta + \overline{\theta} = \varphi$, $\theta_z = \frac{1}{2}\varphi_z = -\frac{c}{2}$, $\theta_{\bar{z}} = \frac{1}{2}\varphi_{\bar{z}} = -\frac{\bar{c}}{2}$. And (1.24) and (1.25) become

$$\begin{cases} \phi_{1z} = \begin{pmatrix} 0 & -\dfrac{1}{2}|H|e^{\frac{\omega}{2}} \\ |H|e^{-\frac{\omega}{2}-it} & \dfrac{1}{2}\omega_z \end{pmatrix}\phi_1, \\ \phi_{1\bar{z}} = \begin{pmatrix} \dfrac{1}{2}\omega_{\bar{z}} & -|H|e^{-\frac{\omega}{2}+it} \\ \dfrac{1}{2}|H|e^{\frac{\omega}{2}} & 0 \end{pmatrix}\phi_1, \end{cases} \tag{3.15}$$

$$\begin{cases} \phi_{2z} = \begin{pmatrix} 0 & -\dfrac{1}{2}|H|e^{\frac{\omega}{2}} \\ |H|e^{-\frac{\omega}{2}+it} & \dfrac{1}{2}\omega_z \end{pmatrix}\phi_2, \\ \phi_{2\bar{z}} = \begin{pmatrix} \dfrac{1}{2}\omega_{\bar{z}} & -|H|e^{-\frac{\omega}{2}-it} \\ \dfrac{1}{2}|H|e^{\frac{\omega}{2}} & 0 \end{pmatrix}\phi_2. \end{cases} \tag{3.16}$$

The integrable condition of the system is $\omega_{z\bar{z}} + |H|^2(\frac{1}{2}e^{\omega} - 2e^{-\omega}) = 0$, which can be transformed into sinh–Laplace equation $u_{z\bar{z}} = -2|H|^2 \sinh u$ by setting $u = \omega - \ln 2$. Given a solution ω of the equation, we can obtain a surface with parallel mean curvature vector in $\mathbf{R}^4$ using the solutions of the system (3.15) and (3.16). Furthermore, if we denote the solution of (3.15) by ϕ_t, that is, take $\phi_t = \phi_1$, then $\phi_2 = \phi_{-t}$.

Now we can investigate the character of the surface using the former discussion.

Theorem 3.5. *The surfaces with parallel mean curvature vector in* $\mathbf{R}^4$ *form a family of surfaces* M^t $(t \in \mathbf{R})$. *They can be determined from the following systems*

$$f_z^t = i\phi_{-t}^* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\phi_t, \qquad f_{\bar{z}}^t = i\phi_{-t}^* \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\phi_t, \tag{3.17}$$

where ϕ_t satisfies

$$\begin{cases} \phi_{tz} = \begin{pmatrix} 0 & -\frac{1}{2}|H|e^{\frac{\omega}{2}} \\ |H|e^{-\frac{\omega}{2}-it} & \frac{1}{2}\omega_z \end{pmatrix} \phi_t, \\ \phi_{t\bar{z}} = \begin{pmatrix} \frac{1}{2}\omega_{\bar{z}} & -|H|e^{-\frac{\omega}{2}+it} \\ \frac{1}{2}|H|e^{\frac{\omega}{2}} & 0 \end{pmatrix} \phi_t. \end{cases} \tag{3.18}$$

In detail, we have

1. M^0 (=$M^{k\pi}$) *lies in* $\mathbf{R}^3$ *with constant mean curvature and the immersion* f *can be expressed by*

$$f = \frac{1}{|H|}\left(2\phi_t^{-1}\frac{\partial}{\partial t}\phi_t - n^t\right)_{t=0},$$

where $n^t = ie^{-\frac{\omega}{2}}\phi_t^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\phi_t$ *is the normal vector of the surface in* $\mathbf{R}^3$ *and* ϕ_t *is the solution of the system* (3.18).

2. M^t ($t \neq k\pi$) *lies in* $S^3(r)$ *with constant mean curvature and* M^t *can be expressed by*

$$|H|\sin t(f^t - f_0) = e^{-\frac{\omega}{2}}\phi_{-t}^* \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}\phi_t,$$

where $r = \frac{1}{|H||\sin t|}$ *and* ϕ_t *is the solution of the* (3.18). *In particular,* $M^{\frac{\pi}{2}}$ *is a minimal surface in* $S^3(\frac{1}{|H|})$.

Proof. For the surface with parallel mean curvature vector in $\mathbf{R}^4$, we consider (3.18).

1. When $t = 2k\pi$, $\phi_{-t} = \phi_t := \phi$. (3.18) turns out to be

$$\begin{cases} \phi_{z} = \begin{pmatrix} 0 & -\frac{1}{2}|H|e^{\frac{\omega}{2}} \\ |H|e^{-\frac{\omega}{2}} & \frac{1}{2}\omega_z \end{pmatrix} \phi, \\ \phi_{\bar{z}} = \begin{pmatrix} \frac{1}{2}\omega_{\bar{z}} & -|H|e^{-\frac{\omega}{2}} \\ \frac{1}{2}|H|e^{\frac{\omega}{2}} & 0 \end{pmatrix} \phi. \end{cases} \tag{3.19}$$

And

$$f_z = i\phi^*\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\phi, \qquad f_{\bar{z}} = i\phi^*\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\phi. \tag{3.20}$$

Because of

$$\frac{\partial}{\partial z}\left(e^{-\frac{\omega}{2}}\phi^*\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\phi\right) = 0,$$

we know the surface lies in $\mathbf{R}^3$. And $n = ie^{-\frac{\omega}{2}}\phi^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\phi$ is the normal vector of the surface in $\mathbf{R}^3$. It can be expressed by a Sym-Bobenko type of representation formula of the surface in $\mathbf{R}^3$ [3]. Taking the solutions ϕ_t of (3.18), we get

$$f = \frac{1}{|H|}\left(2\phi_t^{-1}\frac{\partial}{\partial t}\phi_t - n^t\right)_{t=0}.$$

2. Let $t \neq 2k\pi$. Note that from (3.18)

$$\frac{\partial}{\partial z}\left(e^{-\frac{\omega}{2}}\phi_{-t}^*\begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}\phi_t\right) = |H|\sin t f_z^t,$$

and

$$\frac{\partial}{\partial \bar{z}}\left(e^{-\frac{\omega}{2}}\phi^*_{-t}\begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it}\end{pmatrix}\phi_t\right) = |H|\sin t f^t_{\bar{z}}.$$

Therefore

$$|H|\sin t(f^t - f_0) = e^{-\frac{\omega}{2}}\phi^*_{-t}\begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it}\end{pmatrix}\phi_t.$$

This shows that M^t lies in $S^3(r)$, $r = \frac{1}{|H||\sin t|}$, f_0 is the center.

Next we determine the normal vector and mean curvature vector of surface M in $S^3(r)$.

Since $f^t - f_0$ is a normal vector field of the surface, we know that

$$n^t = ie^{-\frac{\omega}{2}}\phi^*_{-t}\begin{pmatrix} -e^{-it} & 0 \\ 0 & e^{it}\end{pmatrix}\phi_t$$

is the normal vector field of the surface in $S^3(r)$.

From (1.30), we get

$$f^t_{z\bar{z}} = \frac{i}{2}|H|e^{\frac{\omega}{2}}\phi^*_{-t}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\phi_t. \tag{3.21}$$

The mean curvature $H^t_{S^3(r)}$ of the surface in $S^3(r)$ is

$$H^t_{S^3(r)} = \frac{1}{2}e^{\omega}\langle n^t, f^t_{z\bar{z}}\rangle = |H|\cos t.$$

This shows that $M^{\frac{\pi}{2}}$ is a minimal surface in $S^3(r)$ if $t = 2k\pi + \frac{\pi}{2}$. □

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